Description of a Particle with Arbitrary Mass and Spin*

D. L. WEAVER, C. L. HAMMER, AND R. H. GOOD, JR.
Institute for Atomic Research and Department of Physics, Iowa State University, Ames, Iowa

(Received 3 February 1964)

A Lorentz covariant description of a particle and antiparticle with spin $s=0, \frac{1}{2}, 1, \cdots$ and finite rest mass is given in this paper. The wave function has $2(2s+1)$ components so that no auxiliary conditions are needed. The basic idea is to postulate the rest-system Hamiltonian and make the Lorentz transformation to the laboratory system. An algorithm which is a generalization of the Foldy-Wouthuysen transformation is found for constructing the laboratory-system Hamiltonian and polarization operators.

I. INTRODUCTION

IN developing descriptions of free particles with spin, one can work in terms of wave functions and equations of motion as was done by Dirac,[1] Fierz and Pauli,[2] Duffin,[3] Kemmer,[4] and Foldy,[5] in terms of representations of the Lorentz group as was done by Bargmann and Wigner,[6] or in terms of state vectors labeled by momentum and polarization as was done by Jacob and Wick.[7] The group theoretical discussion establishes that a particle-antiparticle may have spin $s=0, \frac{1}{2}, 1, \cdots$ and that, for a given momentum, there are $2(2s+1)$ states if the mass is not zero, 2 states if the mass is zero. For a particular type of particle, a description in terms of wave functions is perhaps of more value than an abstract description. The reason is that such a description not only includes the information in the abstract theory but also facilitates many types of calculations and suggests how interactions can be included. For example, in several cases, to find the electromagnetic interaction one takes advantage of the gauge invariance and replaces $\mathbf{p}$ by $\mathbf{p}-e\mathbf{A}$.

Two types of wave equations for arbitrary spin particles have been studied previously. In the first, as exemplified by the Dirac-Pauli-Fierz theory, the wave function has simple Lorentz transformation properties. However, it is difficult to make detailed calculations because there are auxiliary conditions relating the components of the wave function. In the second type, developed by Foldy,[5] there are no redundant components or auxiliary conditions. However it is difficult to construct interactions because the transformation properties of the wave function are complicated.

It is desirable to have a description without auxiliary conditions on the wave function and also with simple Lorentz transformation properties so that reactions between particles with different spins can be easily studied. The purpose of this paper is to show that such a description can be made. A particle of spin s is described by a $2(2s+1)$ component wave function that transforms in a way similar to the Dirac wave function. The wave function components are proportional to certain types of spinors so that it will be possible to make interactions between particles of different spins simply by a contraction of spinor indices. There is a well-defined Hamiltonian and a straightforward expression for the plane wave solutions for a particle of any spin.

The basic idea can be shown by using the Dirac theory as an example. The Lorentz transformation for a Dirac wave function between two coordinate systems with relative velocity $\mathbf{v}$ is

$$\psi'(x')=\exp[-\tfrac{1}{2}\boldsymbol{\alpha}\cdot(\mathbf{v}/v)\operatorname{arctanh}v]\psi(x). \quad (1)$$

For a state of definite momentum $\mathbf{q}$ the transformation from the rest to the lab frame is then

$$\psi_L=\exp[\tfrac{1}{2}\boldsymbol{\alpha}\cdot(\mathbf{q}/q)\operatorname{arctanh}(q/E)]\psi_R, \quad (2)$$

where E is the energy, $(q^2+m^2)^{1/2}$. It has been shown by Good and Rose[8] and by Bollini and Giambiagi[9] that this transformation can be generalized into an operator which when applied to any particle or antiparticle state gives the result

$$\psi_L=(E/m)^{1/2}e^{-iF}\psi_R. \quad (3)$$

Here

$$e^{\pm iF}=\frac{E+m\pm\beta\boldsymbol{\alpha}\cdot\mathbf{p}}{[2E(E+m)]^{1/2}} \quad (4)$$

is the Foldy-Wouthuysen[10] operator, $\mathbf{p}$ is $-i\boldsymbol{\nabla}$, and E is $(p^2+m^2)^{1/2}$. The operator for the sign of the eigenvalue of the Hamiltonian, which is $+1$ for a particle and -1 for an antiparticle, is β in the rest system and H/E in the lab system. These operators are related by

$$H/E=e^{-iF}\beta e^{+iF}. \quad (5)$$

Equations (4) and (5) lead to

$$H=\boldsymbol{\alpha}\cdot\mathbf{p}+\beta m, \quad (6)$$

the usual Dirac Hamiltonian. Thus, the knowledge of

* This research was done in the Ames Laboratory of the U. S. Atomic Energy Commission, Contribution No. 1430.

[1] P. A. M. Dirac, Proc. Roy. Soc. (London) **A155**, 447 (1936).
[2] M. Fierz, Helv. Phys. Acta **12**, 3 (1939); M. Fierz and W. Pauli, Proc. Roy. Soc. (London) **A173**, 211 (1939).
[3] R. J. Duffin, Phys. Rev. **54**, 1114 (1938).
[4] N. Kemmer, Proc. Roy. Soc. (London) **A173**, 91 (1939).
[5] L. L. Foldy, Phys. Rev. **102**, 568 (1956).
[6] V. Bargmann and E. P. Wigner, Proc. Natl. Acad. Sci. **34**, 211 (1948).
[7] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) **7**, 404 (1959).
[8] R. H. Good, Jr. and M. E. Rose, Nuovo Cimento **24**, 864 (1962).
[9] C. G. Bollini and J. J. Giambiagi, Nuovo Cimento **21**, 107 (1961).
[10] L. L. Foldy and S. A. Wouthuysen, Phys. Rev. **78**, 29 (1950).

the transformation property of the wave function and the assignment of β as the rest-system sign operator leads to the Hamiltonian for the system.

This procedure can be generalized to any spin, as justified in detail in the next section, by defining β and $\boldsymbol{\alpha}$ to be the $2(2s+1)$ order matrices

$$\beta=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \boldsymbol{\alpha}=\frac{1}{s}\begin{pmatrix}\mathbf{s} & 0\\ 0 & -\mathbf{s}\end{pmatrix}. \tag{7}$$

There is a generalization of the Foldy-Wouthuysen transformation operator, Eq. (4), which is not in general unitary. The Hamiltonian, polarization, and other physical operators are well defined. Simple formulas for plane-wave solutions are obtained. In the massless limit, obtained simply by setting $m=0$, all but the solutions parallel and antiparallel to the momentum vanish. In the same way as the theory for an electron reduces to the theory of the two-component neutrino with Hamiltonian $\boldsymbol{\sigma}\cdot\mathbf{p}$, the arbitrary spin theory reduces to the $(2s+1)$ order theory with Hamiltonian $\mathbf{s}\cdot\mathbf{p}/s$, coinciding with an earlier treatment of massless particles.[11] The spin zero specialization is the Schrödinger-Klein-Gordon theory in a two-component form. The spin one specialization is the Duffin-Kemmer theory in a six-component form with a new Hamiltonian different from that of Taketani and Sakata.[12]

Jordan and Mukunda[13] also have given a process for constructing $2(2s+1)$ component wave equations by making a unitary transformation of Foldy's equations. Their description is, however, different from the one given here because their transformation is (except for spin one-half) unrelated to the rest-to-lab Lorentz transformation.

II. BASIC EQUATIONS

Consider first the case of nonzero rest mass. Let the wave equation in the rest system be

$$m\beta\psi_R=i\partial\psi_R/\partial t_R\,, \tag{8}$$

where the R subscripts indicate rest system quantities. The solutions can be written in the form

$$\psi_{R\epsilon k}(\mathbf{e},t_R)=v_{R\epsilon k}(\mathbf{e})\exp(-i\epsilon m t_R)\,, \tag{9}$$

where the $v_{R\epsilon k}$ are the solutions of the eigenvalue problems for the rest-system Hamiltonian

$$\beta m v_{R\epsilon k}=\epsilon m v_{R\epsilon k} \tag{10}$$

and the rest-system polarization

$$\beta\mathbf{e}\cdot\mathbf{s}v_{R\epsilon k}=k v_{R\epsilon k}\,. \tag{11}$$

Here ϵ is ±1, $k=s, s-1, \cdots, -s$, and $\mathbf{e}$ is an arbitrary unit vector giving the quantization direction for the polarization.

For Lorentz transformations continuous with the identity

$$x_\alpha'=a_{\alpha\beta}x_\beta \tag{12}$$

(the Greek indices run from 1 to 4 and x_4 is it), the wave function transformation is postulated to be

$$\psi'(x')=\Lambda(\boldsymbol{\tau})\psi(x)\,, \tag{13}$$

where

$$\Lambda(\boldsymbol{\tau})=\begin{pmatrix}\exp(i\boldsymbol{\tau}\cdot\mathbf{s}) & 0\\ 0 & \exp(i\boldsymbol{\tau}^*\cdot\mathbf{s})\end{pmatrix}, \tag{14}$$

and $\boldsymbol{\tau}$ is a three vector with complex components. For a space rotation through an angle θ in the right-hand sense about the direction $\boldsymbol{\theta}/\theta$ the parameter $\boldsymbol{\tau}$ is $\boldsymbol{\theta}$ itself and the transformation reduces to

$$\psi'(x')=\exp(i\boldsymbol{\theta}\cdot\mathbf{s})\psi(x)\,. \tag{15}$$

For a pure Lorentz transformation, the primed axes having velocity $\mathbf{v}$ relative to the unprimed, $\boldsymbol{\tau}$ is $i(\mathbf{v}/v)$ arctanh$v$ and the transformation becomes

$$\psi'(x')=\exp[-s\boldsymbol{\alpha}\cdot(\mathbf{v}/v)\mathrm{arctanh}v]\psi(x)\,. \tag{16}$$

It is known that this is the correct transformation rule for a Dirac spin $\frac{1}{2}$ wave function but one may ask if it is sensible also for other spins. It is clear that there is a one-to-one correspondence between these transformations and Lorentz transformations. This correspondence is preserved when transformations are applied successively. To see this one simplifies the product

$$\Lambda(\boldsymbol{\tau}_A)\Lambda(\boldsymbol{\tau}_B)=\Lambda(\boldsymbol{\tau}_C) \tag{17}$$

by using Hausdorff's[14] theorem

$$e^Ae^B=e^{A+B+\frac{1}{2}[A,B]_-+\cdots}, \tag{18}$$

where only higher order commutators appear in the exponent. The calculation of $\boldsymbol{\tau}_C$ depends only on the commutators of the spin matrices which are the same for all spins. Since the correspondence is preserved for spin $\frac{1}{2}$, it follows that it is preserved for all spins.

The transformation rule of Eq. (13) can be understood also from the point of view of spinor analysis. A symmetric spinor with $2s$ lower dotted indices gives an irreducible representation of the Lorentz group. It has $2s+1$ independent components which, when reorganized into matrix form, transform according to the rule

$$\chi'(x')=[\exp(i\boldsymbol{\tau}\cdot\mathbf{s})]\chi(x). \tag{19}$$

The spinor defined by

$$\varphi=(C_s\chi)^*\,, \tag{20}$$

[11] C. L. Hammer and R. H. Good, Jr., Phys. Rev. **108**, 882 (1957) and **111**, 342 (1958).

[12] M. Taketani and S. Sakata, Proc. Phys. Math. Soc. Japan **22**, 757 (1939).

[13] T. F. Jordan and N. Mukunda, Phys. Rev. **132**, 1842 (1963).

[14] F. Hausdorff, Leipzig, Ber. Ges. Wiss., Math-Phys. Kl. **58**, 19 (1906).

TABLE I. The transformation operators S and S^{-1}.

Spin	S	S^{-1}
0	1	1
$\frac{1}{2}$	$[E+m+\boldsymbol{\alpha}\cdot\mathbf{p}\beta][2m(E+m)]^{-1/2}$	$(m/E)[E+m-\boldsymbol{\alpha}\cdot\mathbf{p}\beta][2m(E+m)]^{-1/2}$
1	$[m(E+m)+(E+m)\boldsymbol{\alpha}\cdot\mathbf{p}\beta+(\boldsymbol{\alpha}\cdot\mathbf{p})^2][m(E+m)]^{-1}$	$[(E+m)(2E^2-m^2)-m(E+m)\boldsymbol{\alpha}\cdot\mathbf{p}\beta-(2E+m)(\boldsymbol{\alpha}\cdot\mathbf{p})^2][(E+m)(2E^2-m^2)]^{-1}$
$\frac{3}{2}$	$[-(E+m)(E-5m)-(E-13m)\boldsymbol{\alpha}\cdot\mathbf{p}\beta+9(\boldsymbol{\alpha}\cdot\mathbf{p})^2$ $+9(E+m)^{-1}(\boldsymbol{\alpha}\cdot\mathbf{p})^3\beta][32m^3(E+m)]^{-1/2}$	$[(E+m)(18E^2-mE-13m^2)-(54E^2-Em-41m^2)\boldsymbol{\alpha}\cdot\mathbf{p}\beta-9(2E+m)(\boldsymbol{\alpha}\cdot\mathbf{p})^2$ $+9(E+m)^{-1}(6E+5m)(\boldsymbol{\alpha}\cdot\mathbf{p})^3\beta]m^{1/2}[32(E+m)]^{-1/2}[E(4E^2-3m^2)]^{-1}$

where C_s is a unitary matrix with the property

$$C_s\mathbf{s}=-\mathbf{s}^*C_s \tag{21}$$

transforms like

$$\varphi'(x')=[\exp(i\boldsymbol{\tau}^*\cdot\mathbf{s})]\varphi(x). \tag{22}$$

This corresponds to a symmetric spinor with $2s$ upper undotted indices in the usual notation. In these terms the wave function consists of a lower dotted spinor and an upper undotted spinor

$$\psi=\begin{pmatrix}\chi\\ \varphi\end{pmatrix}, \tag{23}$$

a direct generalization of the Dirac wave function.

Given the rest-frame wave functions of Eq. (9), the laboratory-frame wave functions for a particle or antiparticle with momentum $\mathbf{q}$ are, from Eq. (16),

$$\begin{aligned}\psi_{\epsilon k}&=\exp[s\boldsymbol{\alpha}\cdot(\mathbf{q}/q)\operatorname{arctanh}(q/E)]\psi_{R\epsilon k}\\&=\exp[s\boldsymbol{\alpha}\cdot(\mathbf{q}/q)\operatorname{arctanh}(q/E)]v_{R\epsilon k}\exp(i\epsilon q_\alpha x_\alpha).\end{aligned} \tag{24}$$

The most general laboratory-system wave function is found by integrating over all momenta and summing over ϵ and k. The result is

$$\begin{aligned}\psi(\mathbf{x},t)=(2\pi)^{-3/2}m^s\int d\mathbf{p}E^{-1}\sum_{\epsilon k}A_{\epsilon k}(\mathbf{p})\\ \times\exp[s\epsilon\boldsymbol{\alpha}\cdot(\mathbf{p}/p)\operatorname{arctanh}(p/E)]v_{R\epsilon k}\\ \times\exp[i(\mathbf{p}\cdot\mathbf{x}-\epsilon Et)],\end{aligned} \tag{25}$$

where $A_{\epsilon k}(\mathbf{p})$ are arbitrary coefficients. The factor of m^s is included for convenience in taking the limit of zero rest mass. Also $\mathbf{q}$ has been replaced by $\epsilon\mathbf{p}$ so that the symbol $\mathbf{p}$ can be used for both the operator $-i\boldsymbol{\nabla}$ and its eigenvalue.

III. GENERALIZED FOLDY-WOUTHUYSEN TRANSFORMATION

Equation (25) may be written in the form

$$\psi(\mathbf{x},t)=m^sE^{-1/2}S_t\phi(\mathbf{x},t), \tag{26}$$

where S_t is defined by

$$S_t=\exp[s\epsilon\boldsymbol{\alpha}\cdot(\mathbf{p}/p)\operatorname{arctanh}(p/E)]. \tag{27}$$

Here $\mathbf{p}$, E, and ϵ are now the operators $-i\boldsymbol{\nabla}$, $(p^2+m^2)^{1/2}$, and $(i\partial/\partial t)/|i\partial/\partial t|$. The function ϕ has the form

$$\phi(\mathbf{x},t)=(2\pi)^{-3/2}\int d\mathbf{p}E^{-1/2}\sum_{\epsilon k}A_{\epsilon k}(\mathbf{p})v_{R\epsilon k}\times\exp[i(\mathbf{p}\cdot\mathbf{x}-\epsilon Et)]. \tag{28}$$

It is clear that ϕ satisfies the equation

$$E\beta\phi=i\partial\phi/\partial t \tag{29}$$

and so is Foldy's wave function for a particle of spin s. This shows that Foldy's wave function is related to the rest system wave function of the particle.

The exponential in Eq. (27) can be simplified to a finite sum,

$$S_t=\sum_{n=0}^{2s}c_n(\boldsymbol{\alpha}\cdot\mathbf{p}/p)^n\epsilon^n, \tag{30}$$

where c_n depends on p alone, because $\boldsymbol{\alpha}\cdot\mathbf{p}$ satisfies a characteristic equation of degree $2s+1$,

$$\prod_{k=-s}^{s}[s\alpha\cdot(\mathbf{p}/p)-k]=0. \tag{31}$$

For any specific spin the $2s+1$ coefficients c_n can be found from the $2s+1$ independent equations obtained by diagonalizing $\boldsymbol{\alpha}\cdot\mathbf{p}$ in Eqs. (27) and (30).

The time-independent operator

$$S(\mathbf{p})=\sum_{n=0}^{2s}c_n(\boldsymbol{\alpha}\cdot\mathbf{p}/p)^n\beta^n, \tag{32}$$

obtained by replacing ϵ by β in Eq. (30), is equivalent to S_t when it operates on ϕ so that

$$\psi(\mathbf{x},t)=m^sE^{-1/2}S\phi(\mathbf{x},t). \tag{33}$$

This is the generalized Foldy-Wouthuysen[10] transformation operator in the sense that it converts a rest-system function into a laboratory-system function.

Some special cases for S are given in Table I. For spin $\frac{1}{2}$ one sees that S is $(E/m)^{1/2}e^{-iF}$, where e^{iF} is the Foldy-Wouthuysen transformation matrix. In general $m^sE^{-1/2}S$ is not a unitary matrix as it is for spin $\frac{1}{2}$. This is related to the type of invariant integral appropriate for the various spins as discussed in Sec. VI.

Jordan and Mukunda[13] have given a different generalization of the Foldy-Wouthuysen transformation for all spins. Their transformation is designed to produce a

TABLE II. Hamiltonian operators H.

Spin	H
0	βE
$\frac{1}{2}$	$\boldsymbol{\alpha}\cdot\mathbf{p}+m\beta$
1	$[(2E^2-m^2)\beta+2E\boldsymbol{\alpha}\cdot\mathbf{p}-2(\boldsymbol{\alpha}\cdot\mathbf{p})^2\beta]E(2E^2-m^2)^{-1}$
$\frac{3}{2}$	$[(9E^2-7m^2)m\beta+2(13E^2-10m^2)\boldsymbol{\alpha}\cdot\mathbf{p}$ $-9m(\boldsymbol{\alpha}\cdot\mathbf{p})^2\beta-18(\boldsymbol{\alpha}\cdot\mathbf{p})^3][2(4E^2-3m^2)]^{-1}$

generalization of the Foldy-Wouthuysen position operator. It is unitary and is unrelated to the Lorentz transformation except for the spin one-half particle.

To calculate the inverse of S one observes that it also must have the form

$$S^{-1}=\sum_{n=0}^{2s} d_n(\boldsymbol{\alpha}\cdot\mathbf{p}/p)^n\beta^n. \tag{34}$$

The condition $SS^{-1}=1$, together with Eq. (31), gives a set of linear equations from which the coefficients d_n can be found. The inverses for $s\leq 3/2$ are given in Table I.

The result of these considerations is the similarity transformation

$$\tilde{B}=E^{-1/2}SBS^{-1}E^{1/2}. \tag{35}$$

For spin $\frac{1}{2}$ this coincides with the Foldy-Wouthuysen transformation and in general it is to be interpreted as the transformation from the rest to the laboratory system.

IV. HAMILTONIAN AND POLARIZATION OPERATORS

In the rest frame the sign operator β and the polarization $\beta\mathbf{s}$ are the operators of interest. The sign operator H/E and the polarization in the laboratory are found by making the similarity transformation.

For the sign operator one writes

$$H/E=S\beta S^{-1}. \tag{36}$$

It is clear that H is the Hamiltonian since Eqs. (29) and (33) yield

$$H\psi=i\partial\psi/\partial t. \tag{37}$$

The Hamiltonians for $s\leq 3/2$ are given in Table II.

Using the sign operator H/E, one can express S^{-1} in the alternate form

$$S^{-1}=\sum_{n=0}^{2s}(-1)^n c_n(\boldsymbol{\alpha}\cdot\mathbf{p}/p)^n(H/E)^n, \tag{38}$$

where the c_n are the same coefficients as in Eq. (30). To prove this one notes that S_t^{-1} is found by replacing $\mathbf{p}$ by $-\mathbf{p}$ in Eq. (30) and that ϵ can be replaced by H/E when it operates on ψ.

The polarization operator $\mathbf{O}$ is defined by

$$\mathbf{O}=S\beta\mathbf{s}S^{-1}. \tag{39}$$

From the properties of β and $\mathbf{s}$ it follows that

$$[\mathbf{O},H]_-=0, \tag{40}$$

$$[O_i,O_j]_-=i\epsilon_{ijk}(H/E)O_k. \tag{41}$$

For spin $\frac{1}{2}$, $2\mathbf{O}$ is the usual three-vector polarization operator of Dirac theory.[15] For higher spin the detailed formulas for $\mathbf{O}$ are complicated. For example for spin 1 one finds

$$\mathbf{O}=[m(E^2-m^2)(2E^2-m^2)]^{-1}\{m(m^2\beta\mathbf{s}\cdot\mathbf{p}+2E\mathbf{s}\cdot\mathbf{p}\boldsymbol{\alpha}\cdot\mathbf{p})\mathbf{p} -E^3\mathbf{p}\times(\mathbf{p}\times\beta\mathbf{s})-iE(E^2-m^2)[\mathbf{p}\times\mathbf{s},\,\mathbf{p}\cdot\mathbf{s}]_+ -E^2[\mathbf{p}\times(\mathbf{p}\times\mathbf{s}),\,\boldsymbol{\alpha}\cdot\mathbf{p}]_+-i(E^2-m^2)^2\mathbf{p}\times\boldsymbol{\alpha}\}, \tag{42}$$

where $[\ ]_+$ denotes the anticommutator. However in general the operator $\mathbf{O}\cdot\mathbf{p}$ has the simple form

$$\mathbf{O}\cdot\mathbf{p}=(H/E)\mathbf{s}\cdot\mathbf{p}. \tag{43}$$

This result follows from Eq. (36) and the fact that $\mathbf{s}\cdot\mathbf{p}$ commutes with S.

The functions $\psi_{\epsilon k}$ of Eq. (24) are the simultaneous eigenfunctions of H and $\mathbf{O}\cdot\mathbf{e}$,

$$H\psi_{\epsilon k}=\epsilon E\psi_{\epsilon k}, \tag{44}$$

$$\mathbf{O}\cdot\mathbf{e}\psi_{\epsilon k}=k\psi_{\epsilon k}, \tag{45}$$

as can be seen from Eqs. (10) and (11). One can replace $\mathbf{e}$ by $\mathbf{p}/p$ to get eigenfunctions of $\mathbf{O}\cdot\mathbf{p}$.

V. LORENTZ COVARIANCE

Consider the wave function of Eq. (25) rewritten in the form

$$\psi(\mathbf{x},t)=(2\pi)^{-3/2}m^s\int d\mathbf{p}E^{-1}\sum_{\epsilon k}A_{\epsilon k}(\mathbf{p}) \times\Lambda[-i\epsilon(\mathbf{p}/p)\operatorname{arctanh}(p/E)]v_{R\epsilon k} \times\exp[i(\mathbf{p}\cdot\mathbf{x}-\epsilon Et)] \tag{46}$$

and a proper Lorentz transformation as given by Eqs. (12) and (13). The covariance of the theory can be demonstrated by showing that the wave function in the primed system has the same form.

A property of Lorentz transformations is

$$\Lambda(\boldsymbol{\tau})\Lambda[-i\epsilon(\mathbf{p}/p)\operatorname{arctanh}(p/E)] =\Lambda[-i\epsilon(\mathbf{p}'/p')\operatorname{arctanh}(p'/E')]\Lambda(\boldsymbol{\theta}). \tag{47}$$

For the special case of pure rotations, $\boldsymbol{\theta}$ is $\boldsymbol{\tau}$ and for the special case of pure Lorentz transformations,

$$\boldsymbol{\theta}=2\epsilon\frac{\mathbf{p}\times\mathbf{v}}{|\mathbf{p}\times\mathbf{v}|}\arctan\frac{|\mathbf{p}\times\mathbf{v}|}{[1+(1-v^2)^{1/2}](E+m)-\epsilon\mathbf{p}\cdot\mathbf{v}}. \tag{48}$$

Equation (47) can be easily verified for spin $\frac{1}{2}$. The validity for all spins then follows from Eq. (18). Also one may write

$$\Lambda(\boldsymbol{\theta})v_{R\epsilon k}=\sum_l v_{R\epsilon l}\mathfrak{D}_{lk}^{(s)}(\boldsymbol{\theta}). \tag{49}$$

[15] D. M. Fradkin and R. H. Good, Jr., Rev. Mod. Phys. **33**, 343 (1961).

On combining Eqs. (13), (47), and (49) and changing integration variable from $\mathbf{p}$ to $\mathbf{p}'$ according to $p_\mu'=a_{\mu\nu}p_\nu$, one finds

$$\psi'(\mathbf{x}',t')=(2\pi)^{-3/2}m^s\int d\mathbf{p}'E'^{-1}\sum_{\epsilon l} A_{\epsilon l}'(p') \times\Lambda[(-i\epsilon(\mathbf{p}'/p')\operatorname{arctanh}(p'/E')]v_{R\epsilon l} \times\exp[i(\mathbf{p}'\cdot\mathbf{x}'-\epsilon E't')], \quad (50)$$

where

$$A_{\epsilon l}'(\mathbf{p}')=\sum_k \mathfrak{D}_{lk}^{(s)}(\boldsymbol{\theta})A_{\epsilon k}(\mathbf{p}). \quad (51)$$

This demonstrates the covariance of the theory and gives the transformation rule for A.

For space reflection, combined inversion, and time reflection,

$$x_i'=-x_i,\quad t'=t; \quad (52a,b)$$

$$x_i'=x_i,\quad t'=-t, \quad (52c)$$

let the corresponding wave function transformations be

$$\psi'(x')=i\beta\psi(x), \quad (53a)$$

$$\psi'(x')=i\beta[C\psi(x)]^*, \quad (53b)$$

$$\psi'(x')=\gamma_5\beta[C\psi(x)]^*, \quad (53c)$$

where, in the representation of $\boldsymbol{\alpha}$ and β given in Eq. (7),

$$C=\begin{pmatrix}0 & C_s\\ -C_s & 0\end{pmatrix}, \quad (54)$$

$$\gamma_5=\begin{pmatrix}-1 & 0\\ 0 & +1\end{pmatrix}. \quad (55)$$

The factors of i are included in Eqs. (53) to make the charge conjugation defined below covariant. Also factors of ± 1 would be needed to specify the relative parities and time reflection properties of the particles; these are disregarded here since only the free particles are under discussion. The C matrix has the properties

$$C^\dagger C=1, \quad (56)$$

$$C\boldsymbol{\alpha}C^{-1}=\boldsymbol{\alpha}^*, \quad (57)$$

$$C\beta C^{-1}=-\beta^*, \quad (58)$$

$$C\mathbf{s}C^{-1}=-\mathbf{s}^*, \quad (59)$$

$$C\gamma_5C^{-1}=-\gamma_5^*, \quad (60)$$

$$C\Lambda(\boldsymbol{\tau})C^{-1}=[\Lambda(\boldsymbol{\tau})]^*. \quad (61)$$

When Eqs. (46), (52), (53) are combined one recovers Eq. (50) with the following transformation rules for A:

$$A_{\epsilon k}'(\mathbf{p}')=i\epsilon A_{\epsilon k}(\mathbf{p}), \quad (62a)$$

$$A_{\epsilon k}'(\mathbf{p}')=i(-1)^{s-\epsilon k+1}[A_{-\epsilon,k}(-\mathbf{p})]^*, \quad (62b)$$

$$A_{\epsilon k}'(\mathbf{p}')=(-1)^{s-\epsilon k}[A_{\epsilon,-k}(\mathbf{p})]^*, \quad (62c)$$

where $\mathbf{p}'=-\mathbf{p}$. Here the phases of the functions are chosen so that

$$[Cv_{\epsilon,k}]^*=(-1)^{s+\epsilon k}\epsilon v_{-\epsilon,k}. \quad (63)$$

The theory is therefore covariant with respect to these transformations.

The theory is also invariant to the charge conjugation

$$\psi^C(x)=[C\psi(x)]^*. \quad (64)$$

It follows from Eqs. (58), (60), and (61) that this correspondence is invariant to Lorentz transformations, including the reflections. A property is that

$$[\psi^C(x)]^C=(-1)^{2s+1}\psi(x). \quad (65)$$

This follows from Eq. (54) and the fact that

$$C_s^*C_s=(-1)^{2s}, \quad (66)$$

which can be derived from Eq. (21) and the unitarity of C_s.[16]

For spin greater than zero C is determined within a phase factor by Eqs. (56) to (59). Since these equations are needed for the Lorentz covariance of the charge conjugation, the charge conjugation operation is essentially unique.

The previous discussion applies for spin zero in which case $\mathbf{s}=0$ and $C_s=1$. However for spin zero, since $S=1$, the theory is also invariant to the substitution

$$\tilde{\psi}=\beta\psi. \quad (67)$$

Accordingly, factors of β may be inserted in the definitions of the reflections and charge conjugation. For example a consistent set of assignments replacing Eqs. (53) and (64) is

$$\psi'(x')=\psi(x), \quad (68a)$$

$$\psi'(x')=-[\beta C\psi(x)]^*, \quad (68b)$$

$$\psi'(x')=\gamma_5[\beta C\psi(x)]^*, \quad (68c)$$

$$\psi^C(x)=-[\beta C\psi]^*, \quad (69)$$

where here βC is γ_5. It is shown later that these are the standard assignments of the usual Klein-Gordon theory. With this type of charge conjugation one has $(\psi^C)^C=\psi$ in contrast to Eq. (65).

VI. PHYSICAL ASSIGNMENTS

For each spin, one can define an inner product by

$$(\psi^{(l)},\psi^{(n)})=\int E^{-1}d\mathbf{p}\sum_{\epsilon k} A_{\epsilon k}^{(l)}(\mathbf{p})^*A_{\epsilon k}^{(n)}(\mathbf{p}). \quad (70)$$

It is clear that this defines a positive definite Hilbert space. Furthermore, the inner product is invariant to

[16] U. Fano and G. Racah, *Irreducible Tensorial Sets* (Academic Press Inc., New York, 1959), Appendix C.

TABLE III. The invariant integrals $(\psi^{(l)}\,\psi^{(n)})$.

Spin	Integral
0	$\int d\mathbf{x}\psi^{(l)\dagger}E\psi^{(n)}$
$\frac{1}{2}$	$\int d\mathbf{x}\psi^{(l)\dagger}\psi^{(n)}$
1	$\int d\mathbf{x}\psi^{(l)\dagger}\frac{2E[E^2-(\boldsymbol{\alpha}\cdot\mathbf{p})^2]-m^2E}{m^2(2E^2-m^2)}\psi^{(n)}$
$\frac{3}{2}$	$\int d\mathbf{x}\psi^{(l)\dagger}\frac{9[E^2-(\boldsymbol{\alpha}\cdot\mathbf{p})^2]-7m^2}{2m^2(4E^2-3m^2)}\psi^{(n)}$

the Lorentz transformation given in Eqs. (13), (53a), (68a) in the sense that

$$(\psi'^{(l)},\psi'^{(n)})=(\psi^{(l)},\psi^{(n)}). \tag{71}$$

The proof for the transformations continuous with the identity follows directly from the transformation rules for the $A_{\epsilon k}$ given in Eqs. (51). When a charge conjugation is involved, the result is

$$(\psi'^{(l)},\psi'^{(n)})^*=(\psi^{(l)},\psi^{(n)}). \tag{72}$$

In terms of Foldy's wave function, Eq. (28), the inner product has the form

$$(\psi^{(l)},\psi^{(n)})=\int d\mathbf{x}\phi^{\dagger(l)}(x)\phi^{(n)}(x). \tag{73}$$

In terms of the laboratory system functions $\psi(x)$ the general form is

$$(\psi^{(l)},\psi^{(n)})=m^{-2s}\int dx\psi^{(l)\dagger}E(S^{-1})^\dagger S^{-1}\psi^{(n)}. \tag{74}$$

Some special cases are given in Table III.

Given an operator $\mathcal{O}_\psi$ which acts on ψ, there is a corresponding operator $\mathcal{O}_\phi$ which acts on ϕ. The relationship between them is seen from Eq. (33) to be

$$\mathcal{O}_\phi=S^{-1}E^{1/2}\mathcal{O}_\psi E^{-1/2}S. \tag{75}$$

It follows from Eqs. (74) and (75) that, if $\mathcal{O}_\phi$ is Hermitian in the usual sense

$$\int d\mathbf{x}(\mathcal{O}_\phi\phi^{(l)})^\dagger\phi^{(n)}=\int d\mathbf{x}\phi^{\dagger(l)}\mathcal{O}_\phi\phi^{(n)}, \tag{76}$$

then $\mathcal{O}_\psi$ is Hermitian in the sense that

$$(\mathcal{O}_\psi\psi^{(l)},\psi^{(n)})=(\psi^{(l)},\mathcal{O}_\psi\psi^{(n)}). \tag{77}$$

For example H/E and $\mathbf{O}$ are Hermitian by this definition since β and $\beta\mathbf{s}$ are Hermitian in the usual sense. However, $\mathbf{O}$ is not Hermitian in the sense of Eq. (76).

The infinitesimal displacement operators $\mathcal{O}_\psi$ found from the transformation rules for $\psi(x)$ are $\mathbf{p}$, H,

$$\mathbf{J}=\mathbf{x}\times\mathbf{p}+\mathbf{s}, \tag{78}$$

$$\mathbf{G}_\psi=\mathbf{x}H-t\mathbf{p}-is\boldsymbol{\alpha}. \tag{79}$$

For space and time displacements the assumption that $\psi'(x')=\psi(x)$ leads to the operators $\mathbf{p}$ and H. The operators $\mathbf{J}$ and $\mathbf{G}_\psi$ follow directly from Eqs. (15) and (16). In particular for a pure Lorentz transformation with infinitesimal relative velocity $\mathbf{v}$ one finds

$$\psi'(x)=(1-i\mathbf{v}\cdot\mathbf{G}_\psi)\psi(x). \tag{80}$$

The corresponding operators $\mathcal{O}_\phi$ are $\mathbf{p}$, βE, $\mathbf{J}$, and

$$\mathbf{G}_\phi=\tfrac{1}{2}(\mathbf{x}E\beta+E\beta\mathbf{x})-t\mathbf{p}+(m+E)^{-1}\mathbf{p}\times\mathbf{s}\beta. \tag{81}$$

The first three follow directly from Eqs. (36) and (75) and the fact that $\mathbf{p}$ and $\mathbf{J}$ commute with $E^{-1/2}S$. The expression for $\mathbf{G}_\phi$ can be derived from the explicit formula for $\phi(x)$ and the transformation rule for $A_{\epsilon l}(\mathbf{p})$, Eqs. (28) and (51). One writes

$$\phi'(\mathbf{x},t)=(2\pi)^{-3/2}\int d\mathbf{p}'E'^{-1/2}\sum_{\epsilon k}A_{\epsilon k}'(\mathbf{p}')v_{R\epsilon k}\times\exp[i(\mathbf{p}'\cdot\mathbf{x}-\epsilon E't)], \tag{82}$$

substitutes the infinitesimal Lorentz transformation

$$\mathbf{p}'=\mathbf{p}-\mathbf{v}\epsilon E, \tag{83}$$

$$E'=E-\epsilon\mathbf{v}\cdot\mathbf{p}, \tag{84}$$

and expands to first order in $\mathbf{v}$. To this order Eqs. (49) and (51) yield

$$\sum_{\epsilon l}A_{\epsilon l}'(\mathbf{p}')v_{R\epsilon l}=\sum_{\epsilon l}(1+i\boldsymbol{\theta}\cdot\mathbf{s})v_{R\epsilon l}A_{\epsilon l}(\mathbf{p}),$$

where, from Eq. (48),

$$\boldsymbol{\theta}=\epsilon(E+m)^{-1}\mathbf{p}\times\mathbf{v}. \tag{85}$$

The result of all these substitutions is

$$\phi'(x)=(1-i\mathbf{v}\cdot\mathbf{G}_\phi)\phi(x). \tag{86}$$

As expected, since ϕ is Foldy's wave function, these operators are the ones he proposed[5] in order to obtain an appropriate representation of the Lorentz group. One sees that $\mathbf{p}$, βE, $\mathbf{J}$, and $\mathbf{G}_\phi$ are Hermitian in the sense of Eq. (76) so that $\mathbf{p}$, H, $\mathbf{J}$, $\mathbf{G}_\psi$ are Hermitian in the sense of Eq. (77).

The conserved quantities corresponding to the displacement operators are

$$P_i=(\psi,(H/E)p_i\psi)=\int d\mathbf{x}\phi^\dagger\beta p_i\phi, \tag{87}$$

$$P_4=i(\psi,(H/E)H\psi)=i\int d\mathbf{x}\phi^\dagger E\phi, \tag{88}$$

$$\Theta_{ij}=(\psi,(H/E)\epsilon_{ijk}J_k\psi)=\int d\mathbf{x}\phi^\dagger\beta\epsilon_{ijk}J_k\phi\,, \tag{89}$$

$$\Theta_{i4}=-\Theta_{4i}=i(\psi,(H/E)G_{\psi i}\psi)=i\int d\mathbf{x}\phi^\dagger\beta G_{\phi i}\phi\,. \tag{90}$$

The first three are the expectation values of the momentum, energy, and angular momentum and the last gives the center of energy theorem. The justification for these definitions is that P_μ is a Lorentz four vector and that $\Theta_{\mu\nu}$ (with $\Theta_{44}\equiv 0$) is an antisymmetric tensor. The proof of the tensor properties follows from the commutation rules between the operators (these are listed by Foldy[5]) and the fact that the operators are Hermitian. For example, to make the proof that P_μ is a four vector, one again considers the infinitesimal pure Lorentz transformation with relative velocity $\mathbf{v}$. Then it is seen that

$$\begin{aligned}P_i'&=\int d\mathbf{x}\phi'^\dagger\beta p_i\phi'\\ &=\int d\mathbf{x}[(1-i\mathbf{v}\cdot\mathbf{G}_\phi)\phi]^\dagger\beta p_i(1-i\mathbf{v}\cdot\mathbf{G}_\phi)\phi\\ &=P_i+i\mathbf{v}\cdot\int d\mathbf{x}\phi^\dagger\beta[\mathbf{G}_\phi,p_i]\phi\\ &=P_i+iv_iP_4\,,\end{aligned} \tag{91}$$

and similarly

$$P_4'=P_4-i\mathbf{v}\cdot\mathbf{P}, \tag{92}$$

so that P_μ is indeed a four vector. P_μ and $\Theta_{\mu\nu}$ are regular by space reflection and pseudo by time reflection.

Two other conserved quantities are the number of particles

$$N=(\psi,\psi)=\int d\mathbf{x}\phi^\dagger\phi\,, \tag{93}$$

and the charge

$$Q=(\psi,(H/E)\psi)=\int d\mathbf{x}\phi^\dagger\beta\phi\,. \tag{94}$$

VII. SPECIALIZATION TO ZERO MASS

In principle the entire theory presented so far follows from the expression for $\psi(x)$ given in Eq. (25). One can obtain the special case of zero rest mass simply by evaluating this formula for $\psi(x)$ at $m=0$.

Let Eq. (25) be rewritten in the form

$$\psi(x)=(2\pi)^{-3/2}m^s\int d\mathbf{p}E^{-1}\sum_{\epsilon k}\tilde{A}_{\epsilon k}(\mathbf{p})\times S_t v_{R\epsilon k}(\mathbf{p})\exp[i(\mathbf{p}\cdot\mathbf{x}-\epsilon Et)]\,, \tag{95}$$

where S_t is defined in Eq. (27) although with ϵ and $\mathbf{p}$ no longer operators. The different amplitude $\tilde{A}_{\epsilon k}$ and eigenvector $v_{R\epsilon k}(\mathbf{p})$ occur because the quantization direction $\mathbf{e}$ in Eq. (11) has been changed to the $\mathbf{p}$ direction.

Consider a matrix R that commutes with γ_5 and such that

$$R^{-1}\mathbf{s}\cdot(\mathbf{p}/p)R=s_3\,. \tag{96}$$

If a representation with s_3 diagonal is chosen, then, since γ_5 is also diagonal, it follows that $R^{-1}S_tR$ is diagonal. The diagonal elements are

$$[s_3]=(s,s-1,\cdots-s;s,\cdots-s)\,, \tag{97}$$

$$[R^{-1}S_tR]=(e^{\mu s},e^{\mu(s-1)}\cdots e^{-\mu s};e^{-\mu s}\cdots e^{\mu s})\,, \tag{98}$$

where

$$\mu=\epsilon\operatorname{arctanh}(p/E)\,. \tag{99}$$

However $e^{\pm\mu}$ simplifies to

$$e^{\pm\mu}=(E\pm\epsilon p)/m\,. \tag{100}$$

Therefore when the factor of m^s is included only certain of the diagonal elements, corresponding to the $\pm s$ eigenvalues of s_3, survive and one finds

$$[m^sR^{-1}S_tR]_{m=0}=(2p)^s(\tfrac{1}{2}(1+\epsilon),0\cdots0,\tfrac{1}{2}(1-\epsilon);\ \tfrac{1}{2}(1-\epsilon),0\cdots0,\tfrac{1}{2}(1+\epsilon)). \tag{101}$$

This result leads to a considerable simplification in Eq. (95) since

$$\begin{aligned}(m^sS_tv_{R\epsilon k})_{m=0}&=R(m^sR^{-1}S_tR)_{m=0}R^{-1}v_{R\epsilon k}\\ &=(2p)^s[\tfrac{1}{2}(1+\gamma_5)\delta_{k,-s}+\tfrac{1}{2}(1-\gamma_5)\delta_{k,s}]v_{R\epsilon k}\,.\end{aligned} \tag{102}$$

Here use is made of the fact that $R^{-1}v_{R\epsilon k}$ is an eigenfunction of βs_3. The wave function for zero rest mass is therefore

$$[\psi(x)]_{m=0}=2^s(2\pi)^{-3/2}\int d\mathbf{p}p^{s-1}\sum_\epsilon[\tilde{A}_{\epsilon,-s}\tfrac{1}{2}(1+\gamma_5)v_{R\epsilon,-s}+\tilde{A}_{\epsilon,s}\tfrac{1}{2}(1-\gamma_5)v_{R\epsilon,s}]\exp[i(\mathbf{p}\cdot\mathbf{x}-\epsilon Et)]\,. \tag{103}$$

Thus the wave function for arbitrary spin and mass decouples when the mass is zero in the same way as the Dirac wave function for an electron decouples into two two-component neutrino functions.

The detailed connection between the theory for finite mass and the theory for zero mass may be exhibited by showing the connection between the corresponding wave functions. It is known that the wave function for a massless particle with spin s has $2s+1$ components, has a Hamiltonian $\mathbf{s}\cdot\mathbf{p}/s$, and may be written in the form[11]

$$\bar{\psi}=(2\pi)^{-3/2}\int d\mathbf{p}p^{s-1}\sum_\epsilon K_\epsilon(\mathbf{p})u_\epsilon\times\exp[i(\mathbf{p}\cdot\mathbf{x}-\epsilon Et)]\,, \tag{104}$$

where $K_\epsilon(\mathbf{p})$ is a scalar (except for phase factors) amplitude and the functions u_ϵ satisfy

$$\mathbf{p}\cdot\mathbf{s}u_\epsilon=\epsilon psu_\epsilon\,. \tag{105}$$

It is clear that the eigenfunctions are related through

$$v_{R\epsilon,\pm s}=\frac{1}{\sqrt{2}}\begin{pmatrix} u_{\pm\epsilon} \\ \epsilon u_{\pm\epsilon} \end{pmatrix}, \tag{106}$$

so the connection is simply

$$\psi_{m=0}=2^{s-1/2}\begin{pmatrix} \bar{\psi} \\ (C_s\bar{\psi})^* \end{pmatrix}, \tag{107}$$

when an appropriate identification of the K's and $\tilde{A}$'s is made.

VIII. DISCUSSION

The above theory is of course identical with Dirac theory for spin $\frac{1}{2}$. For spin 1 Foldy[5] has shown in detail the connection between the wave function ϕ and the Proca wave function. The relation between the two-component spin zero wave function ψ and the usual Klein-Gordon spin zero wave function ϕ_{KG} is

$$\psi=\frac{1}{\sqrt{2}}\begin{pmatrix} \phi_{\mathrm{KG}} \\ iE^{-1}\partial\phi_{\mathrm{KG}}/\partial t \end{pmatrix}, \tag{108}$$

and between the invariant integrals is

$$(\psi^{(l)},(H/E)\psi^{(n)}) =\frac{i}{2}\int d\mathbf{x}\left(\phi_{\mathrm{KG}}{}^{(l)*}\frac{\partial\phi_{\mathrm{KG}}{}^{(n)}}{\partial t}-\frac{\partial\phi_{\mathrm{KG}}{}^{(l)*}}{\partial t}\phi_{\mathrm{KG}}{}^{(n)}\right), \tag{109}$$

where H/E is simply β for spin zero. Relative to the reflections of Eq. (52) the usual transformation rules for the wave function ϕ_{KG} are

$$\phi_{\mathrm{KG}}'(x')=\phi_{\mathrm{KG}}(x), \tag{110a}$$

$$\phi_{\mathrm{KG}}'(x')=\phi_{\mathrm{KG}}^*(x). \tag{110b,c}$$

Also the charge conjugation is

$$\phi_{\mathrm{KG}}{}^C(x)=\phi_{\mathrm{KG}}^*(x). \tag{111}$$

When these assignments are substituted into Eq. (108) one finds the rules given in Eqs. (68) and (69).

Since the spin zero charge conjugation operation of Eq. (111) or (69) has period two, it is possible to impose the condition

$$\psi^C(x)=\psi(x) \tag{112}$$

and obtain a self-charge-conjugate theory. However as seen from Eq. (65) the charge conjugation operation of Eq. (64) has period two only for half-integer spin. Therefore for integer spin greater than zero one cannot make this theory self-charge-conjugate. The situation is different for the $(2s+1)$ component zero mass theory since it does not have charge conjugation invariance. Also Eq. (65) permits the arguments about superselection rules[17] to be extended to interactions involving high spin particles.

As well as a charge conjugate function, a CP-conjugate function can be defined by

$$\psi^{CP}(\mathbf{x},t)=i\beta[C\psi(-\mathbf{x},t)]^*. \tag{113}$$

It follows from the covariance of the equations of motion relative to the CP transformation, Eqs. (52b) and (53b), that if ψ is a solution then ψ^{CP} is also a solution. It is found that

$$(\psi^{CP})^{CP}=(-1)^{2s}\psi. \tag{114}$$

Accordingly, self CP-conjugate solutions can be chosen for integer spin but not for half-integer spin, just opposite to the charge-conjugation result. This CP-invariance property also applies in the massless limit whereas the C invariance does not. For example, for the photon CP-conjugate states are those of opposite circular polarization and self CP-conjugate states are plane polarized.

Note added in proof: Related papers have been published by S. Weinberg [Phys. Rev. **133**, B1318 (1964) and **134**, B882 (1964)] and by M. M. Saffren [Space Programs Summary No. 37–25, Jet Propulsion Laboratory, California Institute of Technology, 1964 (unpublished), Vol. 4, p. 262].

---

[17] See, for example, P. Roman, *Theory of Elementary Particles* (Interscience Publishers, Inc., New York, 1960), p. 271.